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CONTACT PROBLEM OF PRESSING A STAMP INTO AN ELASTIC HALF-PLANE WITH A THIN REINFORCING COVERING

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The contact problem is investigated when a stamp of sufficiently general configuration is pressed, under the effect of an arbitrary system of forces, into an elastic half-plane with a thin reinforcing layer. It is assumed that the half-plane is in a state of plane strain or generalized plane stress. Relative to the thin reinforcing layer, it is assumed that it bends as an ordinary beam in the vertical direction, while it stretches or is compressed as a bar in the horizontal direction. In other words, the beam bending model, in combination with the model of the uniaxial state of stress of a bar is considered valid for the reinforcing layer. In conformity with the results for thin plates [1, 2], this model has a sufficiently broad range of applicability and the possible error in the magnitudes of the stress which it will admit is ordinarily on the order of the ratio h^2 / l^2 in the case of a layer of finite length l, where h is the height of the layer.

Let us discuss some papers bordering on the problem being investigated and based on the model mentioned.

Let us note that if the reinforcing layer is so flexible that its bending stiffness can be neglected, then this layer will be deformed as a bar in the uniaxial stress state. The uniaxial state of stress model of a bar has been proposed in [3] for application to problems on load transmission from elastic bracing in the form of small thickness gussets (stringers) to massive bodies which are important for engineering practice.

On the other hand, if the deformation of the reinforcing layer as a bar is neglected by considering that it bends only as a beam, then a model is obtained which is well known in problems of the theory of bending of beams and slabs on an elastic basis. Without discussing the results and papers from this area of elasticity theory, let us just note that its fundamental achievements have been examined with sufficient completeness in the monographs [4 —7] and detailed survey papers [8, 9].

Furthermore, contact problems when the stamp is pressed into a plate resting freely on an elastic base have been examined in [10].

Finally, let us mention the paper [11] in which substantially the same problem as in the present paper was examined in a somewhat different formulation. The influence of the reinforcing covering was introduced in the boundary condition for the half-plane, and the solution of the original problem was reduced to the solution of a Fredholm integral equation of the first kind for the unknown pressure under the stamp. An explicit expression for the exact or approximate solution of this equation was not constructed. According to certain results relative to the behavior of the contact stresses near the ends of the stamp, the forces applied to the stamp are transmitted to the base by means of concentrated forces and moments applied to the ends of the contact section, which contradicts the properties of solutions of boundary value problems of elasticity theory and the results of singular integral equations theory. Moreover, the known solutions of classical contact problems for a half-plane with inherent singularities at the ends of the stamp are not obtained from the results in [11] in the absence of the reinforcing covering. In summary, the results obtained in the paper mentioned are false because of the incorrect application of the appropriate mathematical apparatus.

An exact solution of the problem posed is constructed in this paper on the basis of the assumptions presented above. Determination of the distribution laws for the contact stresses under the stamp and under the reinforcing covering reduces to solving integral or integro-differential equations with a Cauchy kernel. These solutions are represented outside the stamp and under the reinforcing coating by using power series, and under the stamp by series in the classical Chebyshev and Jacobi orthogonal polynomials. Numerical results are presented.

1. Formulation of the problem and derivation of the governing equations. Let an elastic half-plane under plane strain conditions, with elastic modulus E_2 and Poisson ratio ν be reinforced at its boundary by a thin covering in the form of a welded or glued infinite elastic layer of small thickness h. Furthermore, let a stamp whose base is characterized by a sufficiently smooth function $f(x)$ (Fig. 1) be pressed into such a half-plane subjected to a vertical force P and a moment M. Determine the distribution laws for the contact stresses under the stamp and the reinforcing covering. It is assumed that only normal contact stresses act under the stamp.

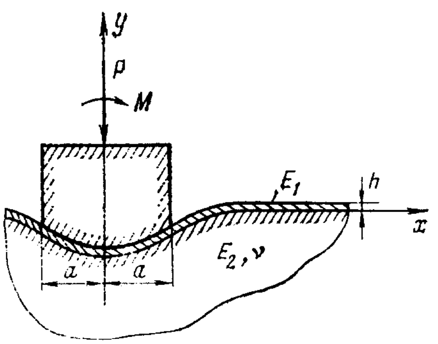


Fig. 1

A beam bending model in combination with the model of the uniaxial state of stress of a bar is used as the fundamental physical model of the reinforcing layer. Since the thickness of the reinforcing layer is assumed small, for the beam to have finite bending stiffness it is necessary that the elastic modulus E_1 of the layer be sufficiently high. Let us consider that E_1 is generally greater than the elastic modulus E_2 of the base.

Let us derive the governing equations of the problem posed.

Let $q(x)$ and $\tau(x)$, respectively, denote the normal and tangential contact stresses applied to the boundary of the elastic half-plane and acting on the lines connecting the reinforcing layer to the base. The pressure of the stamp on the reinforced elastic half-plane is denoted by $p(x)$ and let the contact section be the segment $[-a, a]$, where it is assumed that $a \gg h$. Furthermore, let $u_2(x)$ and $v_2(x)$, respectively, denote the horizontal and vertical displacements of the boundary points of the elastic half-plane due to the mentioned loads $q(x)$ and $\tau(x)$.

Considering the reinforcing layer as a beam in bending, let us write the equation of its elastic axis as

$$D \frac{d^2 v_1}{dx^2} = M(x) \tag{1.1}$$

Here $v_1(x)$ is the vertical deflection of the beam at the section x, $D = E_1 I$ is the beam bending stiffness, I is the moment of inertia of the beam cross section, and $M(x)$ is the bending moment at the section x

$$M(x) = \int_x^\infty (\xi - x)[q(\xi) - p(\xi)]\,d\xi \quad (-\infty < x < \infty) \tag{1.2}$$

Now, considering the reinforcing layer as a bar in the uniaxial stress state, we obtain by using Hooke's law ($\varepsilon_x^{(1)}$ is the axial strain in the x section of the bar)

$$\varepsilon_x^{(1)} = \frac{1}{hE_1}\int_{-\infty}^{x} \tau(\xi)\,d\xi \quad (-\infty < x < \infty) \tag{1.3}$$

The conditions

$$v_1(x) = v_2(x), \quad du_2/dx = \varepsilon_x^{(1)} \quad (-\infty < x < \infty)$$

should hold on the boundary $y = 0$ of the elastic half-plane and, in combination with (1.1), they show that the problem posed is formulated in the form of the relationships

$$du_2/dx = \varepsilon_x^{(1)} \; (-\infty < x < \infty); \quad dv_2/dx = f'(x), \quad |x| < a \tag{1.4}$$

$$D\frac{d^2 v_2}{dx^2} = M(x), \quad |x| > a$$

It follows from this and from the known asymptotic formulas [12] for the displacements for large x that at least

$$q(x) = O(x^{-3}), \quad \tau(x) = O(x^{-3}) \quad \text{for} \quad x \to \infty \tag{1.5}$$

Taking into account the known formulas for $u_2(x)$ and $v_2(x)$ from [12] as well as (1.2) and (1.3) and also that $p(x) \equiv 0$ for $|x| > a$, we represent (1.4) as

$$\int_{-\infty}^{\infty} \frac{\tau(\xi)\,d\xi}{\xi - x} - \vartheta_0 q(x) = \lambda\sigma(x) \quad (-\infty < x < \infty) \tag{1.6}$$

$$\int_{-\infty}^{\infty} \frac{q(\xi)\,d\xi}{\xi - x} + \vartheta_0 \tau(x) = -kf'(x), \quad |x| < a$$

$$\int_{-\infty}^{\infty} \frac{q(\xi)\,d\xi}{\xi - x} + \vartheta_0 \tau(x) = \mu g(x), \quad |x| > a$$

$$\vartheta_0 = \frac{\vartheta_1}{\vartheta_2}, \quad \lambda = \frac{1}{\vartheta_2 h E_1}, \quad k = \frac{1}{\vartheta_2}, \quad \mu = \frac{1}{\vartheta_2 D}$$

$$\vartheta_1 = \frac{(1+\nu)(1-2\nu)}{E_2}, \quad \vartheta_2 = \frac{2(1-\nu^2)}{\pi E_2}$$

The following functions have been introduced here

$$h(x) = \int_x^\infty (x - \xi) q(\xi)\,d\xi, \quad \sigma(x) = \int_{-\infty}^{x} \tau(\xi)\,d\xi \quad (-\infty < x < \infty) \tag{1.7}$$

$$g(x) = \int_{\eta(x)}^{x} h(u)\,du, \quad \eta(x) = -\infty \quad \text{for} \quad x < -a, \quad \eta(x) = \infty \quad \text{for} \quad x > a \tag{1.8}$$

The integrals in the left sides of these equations should be understood in the sense of the Cauchy principal value.

Let us examine two particular cases of the system (1. 6). Let there be no thin reinforcing covering, which is equivalent to compliance with the condition $h = 0$. Then, there results from the last equation of this system that $q(x) \equiv 0$ for $|x| > a$, and from the first equation that $\tau(x) \equiv 0$ for $|x| < \infty$. In this case the system (1. 6) passes into the single equation

$$\vartheta_2 \int_{-a}^{a} \frac{q(\xi)\, d\xi}{\xi - x} = -f'(x), \qquad |x| < a$$

which agrees with the known integral equation of the classical contact problem on pressing of a stamp in an elastic half-space [12 —14].

Now, let the reinforcing layer be so flexible that its bending stiffness can be neglected, i. e. set $D = 0$. Moreover, let a horizontal load of arbitrary intensity be applied to some segment of the upper face of the reinforcing covering instead of pressing a stamp into a reinforced half-plane. For definiteness, let us consider that this load is a horizontal force of magnitude P, concentrated at the point $(0, h)$. Then, as mentioned above, the reinforcing layer will be deformed as a bar in a uniaxial state of stress. In this case, there results from the last equation in the system (1. 6), which holds on the whole axis, that $q(x) \equiv 0$ for $|x| < \infty$. On the other hand, the first equation of the system (1. 6) goes over into the following:

$$\vartheta_2 \int_{-\infty}^{\infty} \frac{\tau(\xi)\, d\xi}{\xi - x} = \frac{1}{hE_1}[\sigma(x) + PH(x)] \qquad (-\infty < x < \infty)$$

where $H(x)$ is the known Heaviside function. The known Melan problem [3] is described by this integro-differential equation.

Therefore, the solution of the problem under consideration reduces to the solution of the system of equations (1. 6), which reduces to known equations in particular cases.

Some conditions resulting from the stamp equilibrium should evidently be appended to the system (1. 6).

It is easy to see that the pressure $p(x)$ under the stamp is determined by the formula

$$p(x) = q(x) + Df^{\mathrm{IV}}(x), \qquad |x| < a \tag{1. 9}$$

Making the constraints imposed on the function $f(x)$ specific, let us require that it has finite derivatives to the fourth order inclusive on the segment, where the first derivative in the interval $(-a, a)$ should satisfy the Hölder condition. Henceforth, some other constraints, which as a set retain sufficiently great arbitrariness for its selection, will be imposed on the function $f(x)$.

Later, only the case of symmetric loading of the stamp will be considered since completely analogous results are obtained in the case of a skew symmetric loading.

It is easy to see that the function $\sigma(x)$ is even in the case under consideration, while the function $g(x)$ is odd.

Now, we obtain an equation on the half-line (a, ∞)

$$\left(1 - \frac{\vartheta_0^2}{\pi^2}\right) q(x) = -\frac{2\mu}{\pi^2} \int_{a}^{\infty} \frac{\xi g(\xi)\, d\xi}{\xi^2 - x^2} + \frac{\lambda \vartheta_0}{\pi^2} \sigma(x) + \frac{2k}{\pi^2} F_+(x) \quad (a < x < \infty) \tag{1. 10}$$

from the last two equations of the system (1. 6) by using the inverse Hilbert transform formula [15] and from the first equation of this system, taking into account the evenness properties of the functions just mentioned.

Here

$$F_+(x) = \int_0^a \frac{\xi f'(\xi)\, d\xi}{\xi^2 - x^2} \tag{1.11}$$

In order to obtain the second analogous equation, let us note that we have from the first equation in (1.6), again by using the inverse Hilbert transform formula

$$\tau(x) = \frac{\vartheta_0^2}{\pi^2}\tau(x) - \frac{\vartheta_0\mu}{\pi^2} g(x) - \frac{\lambda}{\pi^2} r(x), \quad r(x) = \int_{-\infty}^{\infty} \frac{\sigma(\xi)d\xi}{\xi - x} \quad (|x| > a) \tag{1.12}$$

where the third equation in the system (1.6) has been taken into account. It can be shown that

$$r(x) = \vartheta_0 Q(x) + \lambda\Sigma(x), \quad Q(x) = \int_{\eta(x)}^{x} q(\xi)\, d\xi, \quad \Sigma(x) = \int_{\eta(x)}^{x} \sigma(\xi)\, d\xi \tag{1.13}$$

Taking account of (1.12) and (1.13), we obtain the second desired equation, which in combination with (1.10) forms the system of governing equations

$$q(x) = -\lambda_0 \int_a^{\infty} \frac{\xi g(\xi)\, d\xi}{\xi^2 - x^2} + \mu_0\sigma(x) + \nu_0 F_+(x) \tag{1.14}$$

$$\tau(x) = -\lambda_1 g(x) + \mu_0 \int_x^{\infty} q(\xi)\, d\xi + \mu_1 \int_x^{\infty} \sigma(\xi)\, d\xi, \quad a < x < \infty$$

$$\lambda_0 = \frac{2\mu}{\theta}, \quad \mu_0 = \frac{\lambda\vartheta_0}{\theta}, \quad \nu_0 = \frac{2k}{\theta}, \quad \lambda_1 = \frac{\vartheta_0\mu}{\theta}, \quad \mu_1 = \frac{\lambda^2}{\theta}, \quad \theta = \pi^2 - \vartheta_0^2$$

Therefore, in the case of symmetric loading of a stamp the unknown contact stresses $q(x)$ and $\tau(x)$ under the thin reinforcing covering in the range $a < x < \infty$, and therefore, also in the range $-\infty < x < -a$, are determined from the system of singular integro-differential equations (1.14) which contains the functions $g(x)$ and $\sigma(x)$ related to the functions $q(x)$ and $\tau(x)$ by means of (1.7) and (1.8).

Let us proceed to derive the equations from which the unknown contact stresses $q(x)$ and $\tau(x)$ under the reinforcing covering are determined in the interval $(-a, a)$. To this end, let us note that we can write

$$\int_{-a}^{a} \frac{\tau(\xi)d\xi}{\xi - x} - \vartheta_0 q(x) = -\lambda T(x) - f_1(x), \quad T(x) = \int_x^a \tau(\xi)\, d\xi \tag{1.15}$$

$$\int_{-a}^{a} \frac{q(\xi)\, d\xi}{\xi - x} + \vartheta_0\tau(x) = -kf'(x) - f_2(x), \quad -a < x < a$$

$$f_1(x) = \lambda \int_a^{\infty} \tau(\xi)\, d\xi + 2\int_a^{\infty} \frac{\xi\tau(\xi)\, d\xi}{\xi^2 - x^2}, \quad f_2(x) = 2x \int_a^{\infty} \frac{q(\xi)d\xi}{\xi^2 - x^2}$$

on the basis of the first two equations of the system (1.6) and the evenness properties of the functions $q(x)$ and $\tau(x)$

The functions $T(x)$ and $f_1(x)$ are evidently even while the function $f_2(x)$ is odd.

Multiplying the first equation in (1.15) by the imaginary unit i and adding it to the second equation, we arrive after some elementary computations at the single equation

$$\int_{-1}^{1} \frac{\chi_0(s)ds}{s - t} - i\vartheta_0\chi_0(t) = -i\lambda a \int_t^1 \tau_0(s)\, ds - F_0(t) \quad (-1 < t < 1) \tag{1.16}$$

$$\chi_0(t) = q_0(t) + i\tau_0(t), \quad q_0(t) = \frac{aq(at)}{P}, \quad \tau_0(t) = \frac{a\tau(at)}{P}$$

$$F_0(t) = \frac{aF(at)}{P} (|t| < 1); \quad F(x) = kf'(x) + f_2(x) + if_1(x) \quad (|x| < a)$$

On the other hand, the stamp equilibrium condition yields the relationship

$$\int_{-1}^{1} q_0(t)\,dt = A, \quad A = 1 + \frac{2Df'''(a)}{P} \tag{1.17}$$

Therefore, in the case of symmetric loading of a stamp the unknown contact stresses $q(x)$ and $\tau(x)$ under a thin reinforcing covering are determined in the interval $(-a, a)$ after they have been determined in the intervals $(-\infty, -a)$ and (a, ∞) from the system (1. 14), from the singular integro-differential equation (1. 16) which should be considered together with the condition (1. 17).

Let us note that because of the evenness properties of the functions in (1. 16)

$$\overline{\chi_0(-t)} = \chi_0(t), \quad \overline{F_0(-t)} = -F_0(t) \quad (|t| < 1) \tag{1.18}$$

The initial system of equations (1. 6) can be converted also into equations of another kind.

Namely, since the first equation of this system holds on the whole axis, then by using the Fourier transform the function $\tau(x)$ can be expressed in terms of the function $q(x)$. We consequently obtain

$$\tau(x) = -\vartheta_0 \int_{-\infty}^{\infty} L(\xi - x)\, q(\xi)\, d\xi \quad (-\infty < x < \infty) \tag{1.19}$$

$$L(u) = \frac{1}{\pi^2}\left\{\frac{1}{u} - \alpha_1[\operatorname{Ci}(\alpha_1 u)\sin\alpha_1 u - \operatorname{si}(\alpha_1 u)\cos\alpha_1 u]\right\}, \quad \alpha_1 = \frac{\lambda}{\pi}$$

Here $\operatorname{Ci}(x)$ and $\operatorname{si}(x)$ are the cosine and sine integrals, respectively [16].

After substituting the expression for $\tau(x)$ from (1. 19) into the two remaining equations of the system (1. 6), we have

$$\int_{-\infty}^{\infty} \frac{q(\xi)\,d\xi}{\xi - x} - \vartheta_0^2 \int_{-\infty}^{\infty} L(\xi - x)\, q(\xi)\, d\xi = \varphi(x) \quad (-\infty < x < \infty) \tag{1.20}$$

$$\varphi(x) = -kf'(x) \quad \text{при } |x| < a, \quad \varphi(x) = \mu g(x) \quad \text{при } |x| > a$$

Hence, again by using the Fourier transform, we find

$$q(x) = \frac{\mu}{\pi}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) R(\xi - x)\, g(\xi)\, d\xi - F(x), \quad (|x| > a) \tag{1.21}$$

$$R(u) = \frac{A}{u} - \frac{B}{\theta}\{\sin(\beta_1 u)\operatorname{Ci}(\beta_1 u) - \cos(\beta_1 u)[\pi + \operatorname{si}(\beta_1 u)]\}$$

$$A = \frac{\pi}{\theta}, \quad B = \frac{\lambda\pi^2}{\theta} + \lambda, \quad \beta_1 = \frac{\lambda\pi}{\theta}, \quad F(x) = \frac{k}{\pi}\int_{-a}^{a} R(\xi - x)\, f'(\xi)\, d\xi$$

Thus, the contact stresses $q(x)$ under the reinforcing coating can also be determined from the integro-differential equation (1. 21) for $|x| > a$.

In order to obtain the second equation to determine $q(x)$ in the interval $(-a, a)$, let us represent (1. 2) for $|x| < a$ as

$$\int_{-a}^{a} \frac{q(\xi)\,d\xi}{\xi - x} - \vartheta_0^2 \int_{-a}^{a} L(\xi - x)\, q(\xi)\, d\xi =$$

$$- kf'(x) - \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{q(\xi)\,d\xi}{\xi - x} + \vartheta_0^2 \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) L(\xi - x)\, q(\xi)\, d\xi$$

Taking into account the expression presented above for $L(u)$, we arrive at the second equation desired

$$\int_{-1}^{1} \left[\frac{1}{s - t} + \alpha_0 Q_0 (s - t) \right] q_0(s)\, ds = H_0(t) \quad (|t| < 1) \tag{1. 22}$$

$$Q_0(u) = \mathrm{Ci}(a\alpha_1 u) \sin(a\alpha_1 u) - \mathrm{si}(a\alpha_1 u) \cos(a\alpha_1 u), \; q_0(t) = \frac{aq(at)}{P}$$

$$\alpha_0 = \frac{\alpha_1 a \vartheta_0^2}{\theta}, \quad k_0 = \frac{k\pi^2}{\theta}, \quad H_0(t) = \frac{aH_*(at)}{P}$$

$$H_*(x) = -k_0 f'(x) - \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \left[\frac{1}{\xi - x} + \frac{\alpha_0}{a} Q_0 \left(\frac{\xi - x}{a} \right) \right] \times$$
$$q(\xi)\, d\xi \quad (|x| < a)$$

The solution of the integral equation (1. 22) should hence satisfy the condition (1. 17) as well as the condition resulting from the moment equilibrium of the stamp

$$\int_{-1}^{1} t q_0(t)\, dt = A_0, \quad A_0 = \frac{M - 2D\,[af'''(a) - f''(a)]}{aP}$$

Thus, the contact stresses $q(x)$ under a reinforcing covering can be determined in the range $(-a, a)$ after they have been determined for $|x| > a$ from the integro-differential equation (1. 2) as well as from the integral equation (1. 22) whose kernel is represented as the sum of a Cauchy kernel and a continuous kernel generated by the function $Q_0(u)$.

It is important to note that an integral equation relating to the pressure under the stamp $p(x)$ is obtained directly from (1.22) by using (1. 9). This equation evidently has the very same structure as (1.22).

Henceforth, we shall consider mainly only the solutions of the system (1. 14) and of (1. 16) since they are more convenient for the construction of the solution needed in simple form and reach the goal more rapidly.

Let us discuss one important particular case of the problem posed. Since there are no tangential contact stresses under the stamp and the thickness of the reinforcing layer is quite small, it can therefore be asserted with sufficient accuracy that these stresses will be very small quantities also on the lines connecting the layer to the elastic half-plane. Then we will have the contact problem of the pressing of a stamp into an elastic strip of small thickness which lies freely on an elastic base. It is hence assumed that the strip is in contact with the base along its whole length.

Let us write the governing equations in the particular case mentioned. This problem is evidently mathematically equivalent to compliance with the condition

$$\frac{dv_2}{dx} = f'(x), \quad |x| < a; \qquad D\frac{d^2 v_2}{dx^2} = M(x), \quad |x| > a \tag{1. 23}$$

$$v_2(x) = -\vartheta_2 \int\limits_{-\infty}^{\infty} \ln \frac{1}{|x-\xi|} q(\xi)\, d\xi$$

Furthermore, considering just the case of symmetric loading of the stamp, and proceeding perfectly analogously to the exposition above, we find on the basis of (1. 23), that the contact stresses $q(x)$ under the reinforcing covering are determined in the interval (a, ∞) from the singular integro-differential equation

$$q(x) = -\frac{2\mu}{\pi^2} \int\limits_a^{\infty} \frac{\xi g(\xi)\, d\xi}{\xi^2 - x^2} + \frac{2k}{\pi^2} F_+(x) \quad (a < x < \infty) \tag{1. 24}$$

where the function $F_+(x)$ is given by (1. 11). These stresses under the reinforcing covering will be determined in the interval $(-a, a)$ from the integral equation

$$\int\limits_{-1}^{1} \frac{q_0(s)\, ds}{s-t} = -G_0(t) \quad (|t| < 1) \tag{1. 25}$$

$$q_0(t) = aq(at)/P, \quad G_0(t) = aG(at)/P$$

$$G(x) = kf'(x) + 2x \int\limits_a^{\infty} \frac{q(\xi)\, d\xi}{\xi^2 - x^2} \quad (|x| < a)$$

Hence, as before, the solution of the integral equation (1. 25) is obtained after the solution of (1. 24) is known, and should satisfy the condition (1. 17).

Let us note that the integro-differential equation (1. 24) can be reduced to a Fredholm integral equation of the second kind with a continuous kernel. However, we shall not consider this here.

2. Reduction of the fundamental governing equations to infinite systems of linear algebraic equations and their investigation. The method of solving the governing equations obtained in Sect. 1 is based on reducing them to completely, or quasi-completely, regular infinite systems of linear algebraic equations. To this end, let us first turn to the system (1. 14). Starting from (1.5), let us set

$$q(x) = \sum_{n=1}^{\infty} \frac{A_n}{x^{2n+2}}, \quad \tau(x) = \sum_{n=1}^{\infty} \frac{B_n}{x^{2n+1}} \quad (a < x < \infty) \tag{2. 1}$$

where $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are unknown coefficients to be determined. Taking account of relationships (1. 7), (1. 8) and (1. 11), we hence obtain

$$g(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{A_n}{n(4n^2-1)\, x^{2n-1}}, \quad \sigma(x) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{B_n}{nx^{2n}}, \quad (x > a) \tag{2. 2}$$

$$\int\limits_x^{\infty} \sigma(\xi)\, d\xi = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{B_n}{n(2n-1)\, x^{2n-1}}, \quad \int\limits_x^{\infty} q(\xi)\, d\xi = \sum_{n=1}^{\infty} \frac{A_n}{(2n+1)\, x^{2n+1}}$$

$$F_+(x) = -\sum_{n=1}^{\infty} \frac{b_n}{x^{2n}} \quad (x > a); \quad b_n = \int\limits_0^a \xi^{2n-1} f'(\xi)\, d\xi \quad (n = 1, 2, \ldots)$$

Furthermore, let us introduce the function

$$I(x)=\int_a^\infty \frac{\xi g(\xi)\,d\xi}{\xi^2-x^2} \quad (x>a)$$

We will have

$$I(x)=\int_a^x \frac{\xi g(\xi)\,d\xi}{\xi^2-x^2}+\int_x^\infty \frac{\xi g(\xi)\,d\xi}{\xi^2-x^2}=-\frac{1}{x^2}\int_a^x \frac{\xi g(\xi)\,d\xi}{1-\xi^2/x^2}+$$

$$\int_x^\infty \frac{\xi g(\xi)\,d\xi}{\xi^2(1-x^2/\xi^2)}=-\sum_{n=1}^\infty \frac{1}{x^{2n}}\int_a^x \xi^{2n-1}g(\xi)\,d\xi+\sum_{n=1}^\infty x^{2n-2}\int_x^\infty \frac{g(\xi)d\xi}{\xi^{2n-1}}$$

Taking into account the first equation in (2. 2), we find

$$I(x)=\sum_{n=1}^\infty \frac{1}{x^{2n}}\sum_{m=1}^\infty \frac{A_m a^{2n-2m+1}}{2m(4m^2-1)(2n-2m+1)} \quad (x>a) \tag{2. 3}$$

Substituting the expressions (2. 1) —(2. 3) into the system (1. 14) and equating corresponding coefficients, we arrive at a system of an infinite number of linear equations

$$x_m+\frac{\lambda_0 a^3}{2}\sum_{n=1}^\infty K_{m,n}^{(11)}x_n+\frac{\mu_0 a^2}{2}\sum_{n=1}^\infty K_{m,n}^{(12)}y_n=c_m^{(1)} \tag{2. 4}$$

$$y_m+\frac{\mu_1 a^2}{2}\sum_{n=1}^\infty K_{m,n}^{(21)}y_n+\sum_{n=1}^\infty K_{m,n}^{(22)}x_n=c_m^{(2)}, \quad m=1,2,\ldots$$

Here

$$K_{m,n}^{(11)}=[n(4n^2-1)(2m-2n+3)]^{-1} \quad (m,n=1,2,\ldots)$$

$$K_{m,n}^{(12)}=0, \quad n=1,2,\ldots m,\ m+2,\ m+3,\ldots;\ K_{m,n}^{(12)}=(m+1)^{-1}, \quad n=m+1$$

$$K_{m,n}^{(21)}=0, \quad n=1,2,\ldots m,\ m+2,\ m+3,\ldots;\ K_{m,n}^{(21)}=[(m+1)(2m+1)]^{-1}, \quad n=m+1$$

$$K_{m,n}^{(22)}=0, \quad n=1,2,\ldots m-1,\ m+2,\ldots;\quad K_{m,n}^{(22)}=-\mu_0(2m+1)^{-1}, \quad n=m$$

$$K_{m,n}^{(22)}=\lambda_1 a^2[2(m+1)(4m^2+8m+3)]^{-1}, \quad n=m+1$$

$$A_n=a^{2n}x_n, \quad B_n=a^{2n}y_n \quad (n=1,2,\ldots)$$

$$b_{m+1}=a^{2m+1}c_m, \quad c_m=\int_0^1 u^{2m+1}f_u'(au)\,du \qquad (m=0,1,2\ldots)$$

$$c_m^{(1)}=-\nu_0 a c_m, \quad c_m^{(2)}=0 \quad (m=1,2,\ldots)$$

Let us note that the coefficients $\{x_m\}_{m=1}^\infty$ should satisfy the additional relationship

$$\left(\lambda_0 a-\frac{\mu_0\lambda_1}{\mu_1}\right)\frac{x_1}{3}-\lambda_0 a\sum_{n=1}^\infty \frac{x_{n+1}}{(n+1)(4n^2+8n+3)(2n-1)}=-\frac{2\nu_0}{a}c_0 \tag{2. 5}$$

which can be considered as some constraint on the function $f(x)$.

Now, let us turn to the integro-differential equation (1. 16). Its reduction to an infinite system of linear equations is based on the following integral relations:

$$w(x)P_n^{(\gamma-1/2,\,-\gamma-1/2)}(x)+\frac{\operatorname{ctg}\pi\gamma}{\pi}\int_{-1}^1 w(t)P_n^{(\gamma-1/2,\,-\gamma-1/2)}(t)\frac{dt}{t-x}= \tag{2. 6}$$

$$(2\sin\pi\gamma)^{-1}P_{n-1}^{(1/2-\gamma,\ 1/2+\gamma)}(x) \quad (n=1,\ 2,\ \ldots)$$

$$w(x)+\frac{\operatorname{ctg}\pi\gamma}{\pi}\int_{-1}^{1} w(t)\frac{dt}{t-x}=0 \quad (w(x)=(1-x)^{\gamma-1/2}(1+x)^{-\gamma-1/2}) \qquad (2.\ 7)$$

Here $\{P_n^{(\alpha,\ \beta)}(x)\}_{n=0}^{\infty}$ $(\mathrm{Re}\ (\alpha,\ \beta)>-1)$ are Jacobi polynomials [16], which are orthogonal in the segment $[-1,\ 1]$ with the weight $(1-x)^{\alpha}\ (1+x)^{\beta}$.

The validity of (2. 7) follows directly from the fact that the solution of the known V. A. Abramov contact problem [12] is given by the function $w(x)$.

The relationship (2. 6) is also known and follows directly from (16), (20) and (21) in [16], p. 172. These formulas are also contained in [17].

It should be noted that the derivation of the integral relations (2. 6) and (2. 7) is also presented in [18 —20]. (The fact that (2. 6) and (2. 7) had been established earlier in [16 —19] was not apparently known to G. Ia. Popov who obtained equivalent relations in [20].)

Let us represent the integro-differential equation (1. 16) under the condition (1. 17) as

$$\chi_0(t)+\frac{\operatorname{ctg}\pi\gamma}{\pi}\int_{-1}^{1}\frac{\chi_0(s)\,ds}{s-t}=\frac{\lambda a}{\vartheta_0}\int_{t}^{1}\tau_0(s)\,ds-\frac{i}{\vartheta_0}F_0(t) \quad (|t|<1) \qquad (2.\ 8)$$

where we have put $\pi^{-1}\operatorname{ctg}\pi\gamma=i/\vartheta_0$. Furthermore, putting $\gamma=-i\alpha$ we obtain

$$\alpha=\frac{1}{2\pi}\ln\frac{\pi+\vartheta_0}{\pi-\vartheta_0}=\frac{1}{2\pi}\ln(3-4\nu)$$

According to the property of the function $\chi_0(t)$, we can put from (1. 18)

$$\chi_0(t)=[w_0(t)]^{-1}\left[\sum_{n=0}^{\infty}u_nP_{2n}^{(\alpha)}(t)+i\sum_{n=1}^{\infty}v_nP_{2n-1}^{(\alpha)}(t)\right],\quad |t|<1 \qquad (2.\ 9)$$

$$w_0(t)=(1-t)^{1/2+i\alpha}(1+t)^{1/2-i\alpha},\quad P_n^{(\alpha)}(t)=P_n^{(-1/2-i\alpha,\ -1/2+i\alpha)}(t)$$

where $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are unknown real coefficients to be determined.

Before proceeding to the determination of these coefficients, let us present the expression for the integrals

$$K_n(x)=\int_x^1[w_0(y)]^{-1}P_n^{(\alpha)}(y)\,dy \quad (n=0,\ 1,\ 2,\ \ldots) \qquad (2.\ 10)$$

Using (38) from [16], p. 175, we obtain at once

$$K_n(x)=(2n)^{-1}(1-x)^{1/2-i\alpha}(1+x)^{1/2+i\alpha}P_{n-1}^{(1/2-i\alpha,\ 1/2+i\alpha)}(x) \quad (n=1,\ 2,\ \ldots) \qquad (2.\ 11)$$

Then, by the change of variable $y=x+u(1-x)$ $(-1\leqslant u\leqslant 1)$ we evaluate the integral $K_0(x)$ separately. We obtain

$$K_0(x)=\int_x^1(1-y)^{-1/2-i\alpha}(1+y)^{-1/2+i\alpha}\,dy=$$

$$(1-x)^{1/2-i\alpha}(1+x)^{-1/2+i\alpha}\int_0^1(1-u)^{-1/2-i\alpha}(1-uz)^{-1/2+i\alpha}\,du\ \left(z=\frac{x-1}{x+1}\right)$$

Using the known integral representation of the hypergeometric function (see [21], p. 123, formula (1)), we find after some elementary manipulations

$$K_0(x) = (1-2i\alpha)^{-1}(1-x)^{1/2-i\alpha}(1+x)^{1/2+i\alpha}F\left(1,1;\ \frac{3}{2}-i\alpha;\ \frac{1-x}{2}\right) \quad (2.12)$$

Let us note here that condition (1. 17) yields on the basis of (2. 9)

$$u_0 = \frac{A}{\pi}\operatorname{ch}\pi\alpha. \quad (2.13)$$

Now, let us substitute the expression for the function $\chi_0(t)$ from (2. 9) into (2. 8) and let us take account of (2. 6) and (2. 7) as well as (2. 10) —(2. 13). Following the known procedure, we consequently obtain an infinite system of linear equations

$$U_m + \frac{\lambda a \operatorname{sh}\pi\alpha}{4\vartheta_0}(h_{2m}^2 m^{1-\delta})^{-1}\sum_{n=1}^{\infty}\frac{1}{n^\delta}R_{m,n}^{(11)}U_n + \quad (2.14)$$

$$\frac{i\lambda a \operatorname{sh}\pi\alpha}{2\vartheta_0}(h_{2m}^2 m^{1-\delta})^{-1}\sum_{n=1}^{\infty}\frac{n^{1-\delta}}{2n-1}R_{m,n}^{(12)}V_n = d_m^{(1)}$$

$$V_m - \frac{i\lambda a \operatorname{sh}\pi\alpha}{4\vartheta_0}(h_{2m-1}^2 m^{1-\delta})^{-1}\sum_{n=1}^{\infty}\frac{1}{n^\delta}R_{m,n}^{(21)}U_n +$$

$$\frac{\lambda a \operatorname{sh}\pi\alpha}{2\vartheta_0}(h_{2m-1}^2 m^{1-\delta})^{-1}\sum_{n=1}^{\infty}\frac{n^{1-\delta}}{2n-1}R_{m,n}^{(22)}V_n = d_m^{(2)} \quad (m=1,2,\ldots)$$

Here ($\Gamma(x)$ is the Gamma function)

$$u_n = n^{1-\delta}U_n, \quad v_n = n^{1-\delta}V_n \quad (0<\delta<1/2) \quad (n=1,2,\ldots)$$

$$h_m = \sqrt{2}\,(m!)^{-1}\left|\Gamma\left(m+\frac{1}{2}+i\alpha\right)\right|$$

$$d_m^{(1)} = (h_{2m}^2 m^{1-\delta})^{-1}\int_{-1}^{1} C(t)\,w_0(t)P_{2m-1}^{(1/2+i\alpha,\ 1/2-i\alpha)}(t)\,dt$$

$$d_m^{(2)} = (h_{2m-1}^2 m^{1-\delta})^{-1}\int_{-1}^{1} C(t)\,w_0(t)P_{2m-2}^{(1/2+i\alpha,\ 1/2-i\alpha)}(t)\,dt, \quad (m=1,2,\ldots).$$

$$C(t) = -\frac{\lambda a A \operatorname{sh}2\pi\alpha}{2\pi\vartheta_0}\left[(1-2i\alpha)^{-1}\overline{w_0(t)}\,F\left(1,1;\ \frac{3}{2}-i\alpha;\ \frac{1-t}{2}\right) - (1+2i\alpha)^{-1}w_0(t)\,F\left(1,1;\ \frac{3}{2}+i\alpha;\ \frac{1-t}{2}\right)\right] - \frac{2\operatorname{sh}\pi\alpha}{\vartheta_0}F_0(t)$$

$$R_{m,n}^{(j1)} = M_{2m-j,\,2n-1} - N_{2m-j,\,2n-1};\ R_{m,n}^{(j2)} = M_{2m-j,\,2n-2} + N_{2m-j,\,2n-2}, \quad j=1,2$$

$$M_{m,n} = \int_{-1}^{1}(1-t^2)\,P_m^{(1/2+i\alpha,\ 1/2-i\alpha)}(t)\,P_n^{(1/2-i\alpha,\ 1/2+i\alpha)}(t)\,dt$$

$$N_{m,n} = \int_{-1}^{1}(1-t^2)\left(\frac{1-t}{1+t}\right)^{2i\alpha}P_m^{(1/2+i\alpha,\ 1/2-i\alpha)}(t)\,P_n^{(1/2+i\alpha,\ 1/2-i\alpha)}(t)\,dt$$

Therefore, the solution of the integro-differential equation (1. 16) under the condition (1. 17) reduces to the solution of infinite system of linear equations (2. 14).

In the case of the integro-differential equation (1. 24), let us put as before

$$q(x)=\sum_{n=1}^{\infty}\frac{A_n}{x^{2n+2}}\quad (x>a)$$

Then the corresponding infinite system of equations will be

$$x_m+\frac{a^3\mu}{\pi^2}\sum_{n=1}^{\infty}\frac{x_n}{n(4n^2-1)(2m-2n+3)}=\frac{2ak}{\pi^2}c_m\quad (m=1,2,\ldots) \tag{2.15}$$

Its solution should satisfy the condition

$$\mu a^2\sum_{n=1}^{\infty}\frac{x_n}{n(4n^2-1)(3-2n)}=-2kc_0 \tag{2.16}$$

In the case of the integral equation (1.25), let us represent the solution as

$$q_0(t)=(1-t^2)^{-1/2}\sum_{n=0}^{\infty}q_n^{(0)}T_{2n}(t)\quad (|t|<1) \tag{2.17}$$

Here $\{T_n(t)\}_{n=0}^{\infty}$ are Chebyshev polynomials of the first kind.

Furthermore, taking account of the known relationship [16]

$$\int_{-1}^{1}\frac{T_n(s)\,ds}{(s-x)\sqrt{1-s^2}}=\pi U_{n-1}(x)\quad (n=1,2,\ldots)$$

where $\{U_{n-1}(x)\}_{n=1}^{\infty}$ are Chebyshev polynomials of the second kind, and by substituting the expression for $q_0(t)$ from (2.17) into (1.25), we obtain

$$q_n^{(0)}=-\frac{2}{\pi^2}\int_{-1}^{1}G_0(t)\,U_{2n-1}(t)\sqrt{1-t^2}\,dt\quad (n=1,2,\ldots) \tag{2.18}$$

The condition (1.17) hence yields $q_0^{(0)}=A/\pi$, from which the setting of the stamp can be determined as usual [12 — 14].

This remark is not quite exact. We have here in mind only the determination of some constant analogous to that which enters into the known equation of the plane contact problem of pressing a stamp into an elastic half-plane ([13], p. 96, formulas (13) and (14))

$$\int_{-a}^{a}\ln\frac{1}{|x-\xi|}\,p(\xi)\,d\xi=\frac{c-f(x)}{\theta_2}$$

is kept in mind.

If the indefinite constant in the solution of the Flamant problem, yielding the absolutely rigid displacement of a half-plane, is neglected, then the constant c will agree with the constant α (formula (3) on p. 94 in [13]). This last constant can be called the setting of the stamp, and it can be determined by starting from the mentioned Fredholm integral equation of the first kind.

For example, in the case of a stamp with a flat base $f(x)\equiv 0$, we find $c=\alpha=P\theta_2\ln 2/a$ by using the relationship

$$\int_{-a}^{a}\ln\frac{1}{|x-\xi|}\frac{d\xi}{\sqrt{a^2-\xi^2}}=\pi\ln\frac{2}{a}$$

and the stamp equilibrium condition. This same formula is contained in [13] (p. 20, formula (73)), where α should be replaced by α/θ_2.

Namely, the above-mentioned remark should be interpreted in the sense elucidated.

Let us note that after some elementary calculations we will have

$$G_0(t) = \frac{ak}{P} f'(at) + \frac{2}{aP} \sum_{m=0}^{\infty} X_m t^{2m+1}, \quad |t| < 1 \tag{2.19}$$

$$X_m = \sum_{n=1}^{\infty} \frac{x_n}{2m+2n+3} \quad (m = 0, 1, 2, \ldots)$$

After the coefficients $\{x_n\}_{n=1}^{\infty}$ have already been determined from the infinite system (2. 15), the coefficients $\{X_m\}_{m=0}^{\infty}$ will be determined by means of them by using the last formula. Let us note that it is possible to limit oneself to finite series in (2. 19) for specific computations.

It should be noted that the integral equation (1. 22) could be considered in place of the integro-differential equation (1. 16). It is easy to show that the kernel generated by the function $Q_0(t)$ has first partial derivatives square summable in the square $-1 < t$, $s < 1$. On this basis, it can be proved as in [22, 23] that we obtain a quasi-complete infinite system of linear equations in the unknown coefficients by representing the solution of (1. 22) by a series in Chebyshev polynomials of the first kind. However, the singularities of the contact stresses $q_0(t)$ and therefore, the singularities of the pressure under the stamp $p_0(t) = ap(at) / P$ at the ends of the segment $[-1, 1]$ will have the form of a square root instead of the singularities in (2. 9). This last circumstance is explained by the application of different mathematical apparatus, namely, the apparatus of Chebyshev and Jacobi orthogonal polynomials which yield finally the same kinds of singularities (in the form of a square root).

Let us turn to an investigation of the infinite systems of equations obtained. Let us first study the infinite system (2. 4). To this end, the sum

$$S_m = \sum_{n=1}^{\infty} |K_{m,n}^{(11)}| = \sum_{n=1}^{\infty} \frac{1}{|2m-2n+3|\, n(4n^2-1)} \quad (m = 1, 2, \ldots)$$

should be estimated.

Proceeding perfectly analogously to the exposition in [23], it can be shown that at least

$$S_m = O(m^{-1+\varepsilon}) \quad \text{for} \quad m \to \infty \tag{2.20}$$

where ε is some arbitrarily small positive number. Therefore, the infinite system (2. 4) is quasi-completely regular for any values of the parameters λ_0, μ_0, μ_1 and a where the order of the decrease in the corresponding sums is given by (2. 20).

Moreover, it can be shown that the infinite system (2. 4) is completely regular under the condition

$$\max\left\{\frac{\lambda_0 a^3}{6}\zeta(3) + \frac{\mu_0 a^2}{4},\ \frac{\lambda_1 a^3}{60} + \frac{\mu_1 a^3}{12} + \frac{\mu_0}{3}\right\} < 1$$

where $\zeta(s)$ is a Riemann function [21].

Let us note that these results can be refined.

The investigation of the regularity of the infinite system (2. 14) is based on the asymptotic representation formula for Jacobi polynomials [17]

$$P_n^{(\alpha,\beta)}(\cos\varphi) \approx \frac{\cos(N_0\varphi - \beta_0)}{\sqrt{\pi n}(\sin{}^1/_2\,\varphi)^{1/2+\alpha}(\cos{}^1/_2\,\varphi)^{1/2+\beta}} + O(n^{-3/2}), \quad n \to \infty \tag{2.21}$$

$$N_0 = n + \frac{\alpha+\beta+1}{2}, \quad \beta_0 = \frac{2\alpha+1}{4}\pi, \quad 0 < \varphi < \pi$$

In this formula it is usually assumed that α and β are real numbers, where $\alpha,\ \beta > -1$. However, an analysis of the appropriate results from [17] (pp. 222-224) shows that (2. 21) is also valid for the complex α and β satisfying the conditions $\mathrm{Re}\,(\alpha,\ \beta) > -1$. Starting from (2. 21), asymptotic formulas for large m and n can be obtained for the kernels $\{R^{(ij)}_{m,\,n}\}^{\infty}_{m,\,n=1}$ $(i,\ j\ =\ 1,\ 2)$ of the infinite system (2. 14). These formulas have a completely identical structure, and as an illustration, it is enough to limit oneself to one of them, for example, to the kernel $R^{(11)}_{m,\,n}$

$$R^{(11)}_{m,\,n} \approx \frac{4}{\pi\ \sqrt{(2n-1)(2m-1)}}\left[\mathrm{ch}^2\frac{\pi\alpha}{2}\,Q^{(11)}_{m,\,n} - \frac{i\,\mathrm{sh}\,\pi\alpha}{2}\,Q^{(12)}_{m,\,n} + \right. \tag{2. 22}$$

$$\left.\frac{i\,\mathrm{sh}\,\pi\alpha}{2}\,Q^{(21)}_{m,\,n} + \mathrm{sh}^2\frac{\pi\alpha}{2}\,Q^{(22)}_{m,\,n}\right], \quad m,\ n\to\infty$$

$$Q^{(21)}_{m,\,n} = \int\limits_0^\pi \sin 2nu \sin 2mu\psi\,(u)\,du, \quad Q^{(12)}_{m,\,n} = \int\limits_0^\pi \sin 2nu \cos 2mu\psi\,(u)\,du$$

$$Q^{(21)}_{m,\,n} = \int\limits_0^\pi \cos 2nu \sin 2mul\,(u)\,du, \quad Q^{(22)}_{m,\,n} = \int\limits_0^\pi \cos 2nu \cos 2mul\,(u)\,du$$

$$\psi\,(u) = \sin u\,[1 - (\mathrm{tg}\,{}^1\!/_2\,u)^{2i\alpha}], \quad l\,(u) = \sin u\,[1 + (\mathrm{tg}\,{}^1\!/_2\,u)^{2i\alpha}]$$

Now, let us consider the sum

$$S^{(11)}_m = m^\delta \sum_{n=1}^{\infty} \frac{1}{n^\delta}\,|\,R^{(11)}_{m,\,n}\,| \quad (m = 1,\ 2,\ \ldots)$$

By using (2. 22) and the Bessel inequality, it can be shown from the theory of Fourier series [22] that $S_m{}^{11} = O(m^{-1/2+\delta})$ as $m \to \infty$. This same formula holds for the other analogous sums. Therefore, this system is quasi-completely regular for any value of the parameters in the infinite system (2. 14).

It should be noted that a thorough investigation of the basic properties of a biorthogonal system of functions in the space $L_2[-1,1]$ of complex-valued functions generated by a system of polynomials $\{P_n{}^{(1/2-i\alpha,\ 1/2+i\alpha)}\,(t)\}^{\infty}_{n=0}$ permits proof of the complete regularity of the infinite system (2. 14) also for some values of the parameters.

3. Example. Let us here consider in greater detail the above-mentioned particular case of the problem being discussed, when a stamp is pressed into an elastic strip of small thickness which rests freely on an elastic half-plane. Then, in the case of a symmetrically loaded stamp od symmetric configuration, the unknown normal contact stresses $q\ (x)$ under the strip will be determined from (1. 24) and (1. 25), and the pressure $p\ (x)$ under the stamp from (1. 9).

Let us take the function $f\ (x)$ characterizing the base of the stamp in the form of the polynomial

$$f\,(x) = \sum_{p=0}^{N} a_p x^{2p} \tag{3. 1}$$

of arbitrary but fixed degree $2N$ which can be used to approximate beforehand the given function $f\ (x)$ from the above-mentioned class to any accuracy. Fixing the value of the half-length of the contact section between the stamp and the reinforced half-plane in the subsequent discussions, let us set $a = 1$. The infinite system of linear algebraic equations (2. 15) and the additional condition (2.16) which is equivalent to the integro-differential equation (1.24), hence, respectively, become in the case of the function $f(x)$ from (3.1):

$$x_m + \frac{\mu}{\pi^2}\sum_{n=1}^{\infty}\frac{x_n}{n(4n^2-1)(2m-2n+3)} = -\frac{2k}{\pi^2}c_m \quad (m=1, 2, \ldots) \tag{3. 2}$$

$$\mu\sum_{n=1}^{\infty}\frac{x_n}{n(4n^2-1)(3-2n)} = -2kc_0 \tag{3. 3}$$

Here

$$c_m = 2\sum_{p=1}^{N}\frac{pa_p}{2(m+p)+1} \quad (m=0, 1, 2\ldots) \tag{3. 4}$$

The function $q(x)$, the contact stress under the reinforcing covering outside the stamp, will be given by the formula

$$q(x) = \sum_{n=1}^{\infty}\frac{x_n}{x^{2n+2}} \quad (1<x<\infty) \tag{3. 5}$$

continued in an even manner in the interval $(-\infty, -1)$.

According to (3. 4), the solution of the infinite system (3. 2) can be represented by

$$x_m = -\frac{4k}{\pi^2}\sum_{p=1}^{N} pa_p x_m^{(p)} \quad (m=1, 2, \ldots) \tag{3. 6}$$

where $x_m^{(p)}$ $(m = 1, 2, \ldots; p = 1, 2, \ldots N)$ is the solution of the infinite system

$$x_m^{(p)} + \frac{\mu}{\pi^2}\sum_{n=1}^{\infty}\frac{x_n^{(p)}}{n(4n^2-1)(2m-2n+3)} = \frac{1}{2(m+p)+1} \quad (m=1, 2, \ldots; \; p=1, 2, \ldots, N) \tag{3. 7}$$

Condition (3. 3) is written as

$$\frac{\mu}{\pi^2}\sum_{p=1}^{N} pa_p Y_p = \sum_{p=1}^{N}\frac{pa_p}{2p+1}, \quad Y_p = \sum_{n=1}^{\infty}\frac{x_n^{(p)}}{n(4n^2-1)(3-2n)} \quad (p=1, 2, \ldots) \tag{3. 8}$$

Thus, one of the coefficients a_p $(p = 1, 2, \ldots, N)$ must be determined from the relationship (3. 8), and the remaining coefficients will be given arbitrarily. As has already been said, this relationship assures the possibility of representing the function $q(x)$ by (3. 5).

After the solution (3. 6) of the infinite system (3. 2) has been determined, after some algebraic computations, the solution of the integral equation (1. 25) is represented because of (2. 17) — (2. 19) by the formula

$$q(x) = \frac{1}{\sqrt{1-x^2}}\left[\frac{P+2E_1If'''(1)}{\pi} - \sum_{m=1}^{\infty} q_m T_{2m}(x)\right] \quad (-1<x<1) \tag{3. 9}$$

$$q_m = \frac{4}{\pi^2}\sum_{n=0}^{\infty} H_{m,n}S_n \quad (m=1, 2, \ldots)$$

$$H_{m,n} = \int_{-1}^{1} t^{2n+1}\tilde{U}_{2m-1}(t)\sqrt{1-t^2}\,dt \quad (n=0, 1, 2, \ldots; m=1, 2, \ldots)$$

$$S_n = \begin{cases} k(n+1)a_{n+1} + X_n, & n=0, 1, 2, \ldots N-1 \\ X_n, & n=N, N+1, \ldots \end{cases}$$

$$X_n = \sum_{p=1}^{\infty}\frac{x_p}{2n+2p+3}, \quad n=0, 1, 2, \ldots$$

where the coefficients x_p $(p = 1, 2, \ldots)$ are given by (3. 6). Evidently, we should have $q(x) > 0$ for $-1 < x < 1$. There hence results that the force P impressing the stamp cannot be arbitrary and should be subject to some constraint, for example, the following:

$$P + 2E_1 I f'''(1) > \pi \max_{|x| \leqslant 1} \left| \sum_{m=1}^{\infty} q_m T_{2m}(x) \right| \qquad (3.\ 10)$$

We take $h = 0.05$ for the numerical computations. Then the moment of inertia of the cross section of the reinforcing covering $I = bh^3 / 12$, where b is the width of the section (equal to one in the case of plane strain), has the value $I = 10.417 \cdot 10^{-6}$.

Let the elastic half-plane be fabricated from lead with the elastic modulus $E_2 = 0.17 \cdot 10^6$ kg/cm ^{2} and the Poisson's ratio $\nu = 0.42$, and let the reinforcing covering be from the materials, respectively, (a) rolled aluminum with $E_1 = 0.69 \cdot 10^6$ kg/cm^2; (b) aluminum bronze, cast with $E_1 = 1.05 \cdot 10^6$ kg/cm^2; (c) chrome-nickel steel with $E_1 = 2.1 \cdot 10^6$ kg/cm^2 (the Poisson's ratio of the reinforcing layer material plays no part here).

As regards the specific form of the function $f(x)$, by starting from (3. 1) let us consider the three cases $N = 2$, $N = 3$ and $N = 4$. Let us set $a_2 = a_3 = a_4 = 1$ in all these cases. The coefficient a_1 corresponding to these cases will be determined from condition (3. 8) and the coefficient a_0 cannot be given since it does not enter into (3. 9).

The calculations were performed on the "Nairi" electronic computer. According to these calculations, the values of the coefficient a_1 corresponding to the cases mentioned are

a	—2.5542	—6.7000	—12.3565
b	—2.5464	—6.6713	—12.2911
c	—2.5275	—6.6016	—12.1327

A truncated system of equations (3. 7) with eight unknowns was solved by the Gauss method corresponding to the mentioned values of N. Then by using these solutions and the values of the coefficient a_1 just presented, values of the coefficients x_m $(m = 1, 2, \ldots, 8)$ were calculated by means of (3. 6). It hence turns out that as the value of N increases in each of the given cases (a) – (c), these coefficients first grow noticeably and then decrease noticeably. On the other hand, for fixed N, these same coefficients grow sufficiently rapidly in each case. The first of the mentioned regularities is illustrated in the table for case (c).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
N	147.2 469.7 969.5	156.4 530.4 1133.7	166.4 596.7 1328.4	142.1 525.8 1198.4	109.0 411.7 953.7	74.3 285.1 668.3	43.3 168.0 397.4	17.8 69.5 165.7

It should be noted that the influence of the reinforcing layer in the contact stress distribution law (3. 9) is actually due to the coefficients x_m.

Then the coefficients q_m, H_{mn} and S_n in (3. 9) and the values of the function $q(x)$ were calculated by (3. 9) and (3. 5) at some point of their range of variation. The minimal value of P satisfying condition (3. 10) in all the cases discussed above was hence found, namely: $P_{\min} = 3.6140 \cdot 10^6$ kg. Presented below as an illustration are values of the function $q(x)$ calculated by means of (3. 9) and (3. 5) in case (c) for $N = 3$

x	0	0.1	0.3	0.5	0.7
$q(x)$	561525	580890	742616	1105032	1779174

x	1.01	1.02	1.04	1.06	1.08	1.1	1.6	2.1	2.6	3.01	3.6
$q(x)$	2793	2557	2158	1835	1572	1357	124	32	12	6	3

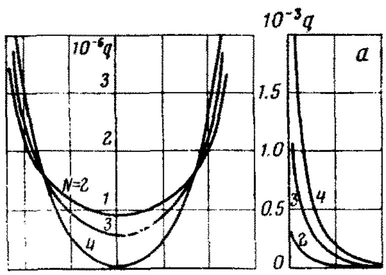
Values of the function $q(x)$ in cases (c), remain quantities of approximately the same order at the same points.

A further analysis of these numerical results shows that the values of the function $q(x)$ grow rapidly, tending to infinity as $x \to 1$ in each separate case when $|x| < 1$ as the argument grows in absolute value. An abrupt drop in the value of this function occurs upon passage through the point $x = 1$.

The values of the function $q(x)$ diminish noticeably in some neighborhood of the middle of the range $(-1,1)$ in each of the cases under consideration as N increases. A reverse regularity, which is extrapolated outside the range $(-1,1)$ being extended to values of the function $q(x)$ for $|x| > 1$, is observed in the neighborhoods of the end-points of this range. When the different cases (a) — (c) are compared for a fixed value of N, then a reverse regularity is observed for values of the function $q(x)$ for $|x| < 1$.

Graphs of the function $q(x)$ corresponding to cases (a) — (c) (Fig. 2) are constructed for the above-mentioned values of N on the basis of the numerical results described. The appropriate parts of the graphs are shown in a magnified scale to represent graphically the history of the change in this function for $|x| > 1$.

Let us note that graphs of contact pressure under the stamp, defined by (3. 9), can easily be constructed.



Let us note that if we set $f(x) \equiv a_0$ by taking account of (3. 1), i. e. we consider the case of a stamp with a flat base, then according to (3. 2) and (3. 4) all the coefficients x_m will be zero, and therefore, the contact stresses under the reinforcing covering outside the stamp will be identically zero. According to (1. 9), on the other hand, $p(x) \equiv q(x)$ ($|x| < 1$),

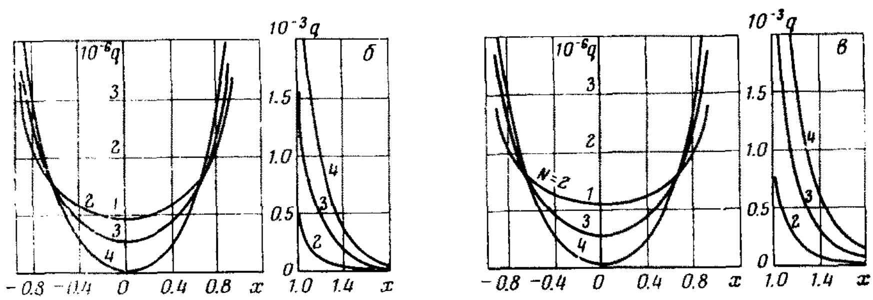


Fig. 2

and (3. 9) yields the known Sadovskii solution in this case. Therefore, within the framework of the assumptions taken, the reinforcing covering exerts no effect on the distribution law of the contact pressure under the stamp in the case of a stamp with a flat base, and moreover, no contact stresses originate under it outside the stamp. However, as the stamp base deviates from the rectilinear, the influence of the reinforcing covering on the contact stress distribution law becomes significant. This fact has been noted above and is shown in the graphs.

In conclusion, let us note that an investigation of the regularities in the change in size of the contact section as a function of the outline of the stamp base, as well as of the elastic and geometric constants of the reinforcing covering and the base, is of interest. We do not, however, perform this investigation here. It can be carried out by a known method [12 — 14].

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CYLINDRICAL BENDING OF A PLATE BY RIGID STAMPS

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A periodic contact problem of the cylindrical bending of a plate by rigid stamps is considered from the aspect of equations of elasticity theory as well as Kirchhoff-Love theory with and without transverse compression of the material in the contact zone taken into account. Analysis of the solutions obtained permits illumination of the question of the error and of the possibility of using the classical theory of plates and shells in analyzing contact problems. A comparative analysis is given of the nature of the distribution and of the magnitude of the stresses of the plate in the contact zone, of the character of the contact reaction distribution and the dependence between the magnitude of the contact zone and the force applied to the stamp. The apparatus of integral equations is used in considering the problem from the aspect of elasticity theory, while the solution is obtained in closed form by means of Kirchhoff theory.

An analogous problem on the basis of the elasticity theory equations has been solved in [1] also by a method different from that elucidated below. However, only sufficiently thick plates (the ratio between the thickness and the characteristic dimension is not less than $^1/_{20}$) are considered there. But a comparison between the stresses obtained when using different theories can yield the most correct answer about the applicability of any theory.

1. Solution on the basis of elasticity theory equations. Let us consider an infinite plate of thickness h (Fig. 1) occupying the xz plane and loaded by a system of rigid stamps. The stamps are identical and arranged with a constant spacing $2l$, have a cylindrical base surface so that the contact occurs over the whole length along